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A Note on Entire Functions Used in Analytic Interpolation

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A NOTE ON ENTIRE FUNCTIONS USED IN ANALYTIC INTERPOLATION

by

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A note on entire functions used in analytic interpolation

1. Introduction.

The functions $F_k(t,u)$ defined by

$$F_k(t,u) = \left\{\frac{\sin \pi u}{\pi}\right\}^k \sum_{n=-\infty}^{\infty} e^{-t\pi(u+n)^2} \frac{(-1)^{nk}}{(u+n)^k}, \tag{1.1}$$

were treated already in the preliminary report R 143 of the Computation Department of the Mathematical Centre and in a paper titled: A class of entire functions used in analytic interpolation. The zeros of $F_k(t,u)$ are all on the lines Re $u = n + \frac{1}{2}$, n being an integer, for k equal to 0, 1 and 2. The author was informed by Prof. I.J. Schoenberg that it is possible to show this fact for all integral values of k.

The object now, however, is in the first place the functions $G_k(t,u)$ defined by

$$G_k(t,u) = \left\{\frac{\sin \pi u}{\pi}\right\}^k \sum_{k=-\infty}^{\infty} e^{-t\pi(u+n)^2} \frac{(-1)^{n(k+1)}}{(u+n)^k}. \tag{1.2}$$

The properties of $G_k(t,u)$ are analoguous to those of $F_k(t,u)$. It is possible to define two other classes of functions, by replacing u by $u+\frac{1}{2}$ (just as is the case with the thêtafunctions), but these classes do not give new results with respect to exponential series. This report gives namely in the second place some inequalities on exponential series, which can be derived from the properties of $F_k(t,u)$ and $G_k(t,u)$.

2. Integral-representations.

One has for $t > 0$

$$G_k(t,u+1) = -G_k(t,u),$$
$$G_k(t,-u) = G_k(t,u),$$
$$G_k(t,u) = -G_k(t,1-u),$$

and $$G_k(t,\tfrac{1}{2}) = 0.$$

So it appears useful to restrict oneself to the strip $0 \leqslant \text{Re } u \leqslant 1$. An asterisk stands for complex conjugated and if $\omega(\sigma)$ has the same meaning as in the papers mentioned above, one can write:

$$G_k(t,u)= 2^{-k} \int_{-k}^{k} \omega(\sigma)d\sigma \sum_{n=-\infty}^{\infty} (-1)^n e^{-t\pi(u+n)^2+\pi i\sigma(u+n)}$$

$$= 2^{-k} .t^{-\frac{1}{2}}.e^{-\pi iu} \int_{-k}^{k} \omega(\sigma)d\sigma \sum_{n=-\infty}^{\infty} e^{2\pi inu-\frac{\pi}{4t}(2n-\sigma-1)^2} .$$

So for real y $G_k(t,iy) > 0$.

Further one has

$$G_k(t,\tfrac{1}{2} + i \tfrac{\lambda}{t})= 2^{-k} e^{\frac{\lambda^2\pi}{t}} \int_{-k}^{k} \omega(\sigma)d\sigma \cdot \sum_{n=-\infty}^{\infty} (-1)^n e^{\{-(n+\frac{1}{2})^2\pi t-\pi i(2n+1)(\lambda-\frac{\sigma}{2})-\frac{\pi\lambda\sigma}{t}\}} \quad (2.1)$$

and

$$G_k(t,\tfrac{1}{2} + i \tfrac{\lambda}{t})= -2^{-k}.i.t^{-\frac{1}{2}} e^{\frac{\pi\lambda}{t}} \int_{-k}^{k} \omega(\sigma)d\sigma \sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{2\pi n\lambda}{t}-\frac{\pi}{4t}(2n-\sigma-1)^2}$$

and so $G_k(t,\frac{1}{2} + \frac{i\lambda}{t})$ is imaginary for real λ .

Introducing the odd entire function

$$\eta(z)= \sum_{n=-\infty}^{\infty}(-1)^n e^{-(n+\frac{1}{2})^2\pi t - 2\pi i(n+\frac{1}{2})z}, \quad (2.2)$$

which is imaginary for real z and satisfies the relations

$$\eta(z+2)= \eta(z),$$
$$\eta(z+1)= -\eta(z),$$

one finds, m being an integer,

$$G_k(t,\tfrac{1}{2}+i \tfrac{2m+1}{2t})=2^{-k+1}(-1)^{m+1} e^{\frac{\pi}{4t}(2m+1)^2}\int_0^k \omega(\sigma)\cosh\left\{\frac{\pi(2m+1)\sigma}{2t}\right\}\eta\left(\frac{1-\sigma}{2}\right)d\sigma \quad (2.3)$$

and

$$G_k(t,\tfrac{1}{2}+i \tfrac{m}{t})= 2^{-k+1}(-1)^m e^{\frac{\pi}{t} m^2} \int_0^k \omega(\sigma)\sinh\left(\frac{\pi m\sigma}{t}\right) \eta\left(\frac{\sigma}{2}\right)d\sigma. \quad (2.4)$$

Formula (2.1) can be rewritten as:

$$G_k(t,\tfrac{1}{2}+i \tfrac{\lambda}{t})= 2^{-k} e^{\frac{\lambda^2\pi}{t}} \int_0^k \omega(\sigma)\, \eta\left(\lambda-\frac{\sigma}{2}\right) e^{-\frac{\pi\lambda\sigma}{t}} d\sigma . \quad (2.5)$$

Let be N(2m) the number of zeros of $G_k(t,u)$ contained in the rectangle $[i(2m-k)/2t,\ 1+i(2m-k)/2t,\ -i(2m-k)/2t,\ 1-i(2m-k)/2t]$, one knows that $N(m)$ is odd for every integer $m \geqslant k$. This number $N(2m)$ can be determined by the method given in section 3 of the already mentioned paper. Looking for the signs in the points $u = \frac{1}{2} + i\,(2m+1)/2t$ and $u = \frac{1}{2} + im/t$ by means of (2.3) and (2.4) one can show the

Theorem: For k equal to not, one or two, all zeros of $G_k(t,u)$ are on the lines $\operatorname{Re} u = n + \frac{1}{2}$, n being an integer and $t \geqslant 0$. For k even, the imaginary part of the zeros approaches the value m/t, m large, while for k odd this part approaches the value $(2m+1)/2t$.

The following productrepresentation holds:

$$G_k(t,u) = P(t).(1-e^{2\pi iu}) \prod_{m=1}^{\infty} (1+e^{2\pi iu-2\pi y_m})(1+e^{-2\pi iu-2\pi y_m}),$$

where $P(t)$ is a function independent of u. The derivation being similar to that of the representation for $F_k(t,u)$ is not given here.

The function $G_k(t,u)$ satisfies the recurrencerelations (7.1) of R 143. The other relations of section 7 of R 143 serve as well for $G_k(t,u)$ as for $F_k(t,u)$. $G_0(t,u)$ is again a thêtafunction.

3. Some inequalities.

The following two inequalities $F_k(t,iy) > 0$ and $G_k(t,iy) > 0$, for real y, give rise to interesting properties.

For k = 1 one computes

$$y \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}}{n^2+y^2} \left[n \sin(2t\pi ny)+y \cos(2t\pi ny)\right] > -\tfrac{1}{2}, \qquad (3.1)$$

and

$$y \sum_{n=1}^{\infty} \frac{e^{-t\pi n^2}}{n^2+y^2} \left[n \sin(2t\pi ny)+y \cos(2t\pi ny)\right] > -\tfrac{1}{2}. \qquad (3.2)$$

So for each integer h, one shows by putting $y = h/2t$

$$h^2 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}}{h^2+4n^2t^2} > -\tfrac{1}{2}. \qquad (3.3)$$

For k = 2, one obtains

$$y^2 \sum_{n=1}^{\infty} \frac{e^{-t\pi n^2}}{(n^2+y^2)^2} \left[(n^2-y^2)\cos(2nt\pi y)-2ny \sin(2nt\pi y)\right] < \tfrac{1}{2}, \qquad (3.4)$$

and

$$y^2 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}}{(n^2+y^2)^2} \left[(n^2-y^2)\cos(2nt\pi y)-2n \sin(2nt\pi y)\right] < \tfrac{1}{2}, \qquad (3.5)$$

and by putting y = h/2t it results that

$$h^2 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}}{(4n^2t^2+h^2)^2} (4\ t^2n^2-h^2) \quad < \tfrac{1}{2}, \tag{3.6}$$

and

$$h^2 \sum_{n=1}^{\infty} \frac{e^{-t\pi n^2}}{(4n^2t^2+h^2)^2} (4\ t^2n^2-h^2) \quad < \tfrac{1}{2}. \tag{3.7}$$

For k = 3, one has

$$y^3 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}}{(n^3-3ny^2)^2+(3n^2y-y^3)^2} \cdot$$

$$\cdot\left[(n^3-3ny^2)\sin(2t\pi ny)+(3n^2y-y^3)\cos(2t\pi ny)\right] > -\tfrac{1}{2}, \tag{3.8}$$

and

$$y^3 \sum_{n=1}^{\infty} \frac{e^{-t\pi n^2}}{(n^3-3ny^2)^2+(3n^2y-y^3)^2} \cdot$$

$$\cdot\left[(n^3-3ny^2)\sin(2t\pi ny)+(3n^2y-y^3)\cos(2t\pi ny)\right] > -\tfrac{1}{2}. \tag{3.9}$$

Putting y = h/2t, one finally finds

$$h^4 \sum_{n=1}^{\infty} (-1)^n \frac{e^{-t\pi n^2}(12n^2t^2-h^2)}{(8n^3t^3-6nh^2t)^2+(12n^2ht^2-h^3)^2} > -\tfrac{1}{2}, \tag{3.10}$$

and

$$h^4 \sum_{n=1}^{\infty} \frac{e^{-t\pi n^2}(12n^2t^2-h^2)}{(8n^3t^3-6nh^2t)^2+(12n^2ht^2-h^3)^2} > -\tfrac{1}{2}. \tag{3.11}$$

It is clear, how to extend this list of formulae.